

Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces: A Study and Proposed Generalizations

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ABSTRACT

Fixed point theory is an important area of nonlinear analysis with applications in differential equations, optimization, approximation theory, control, and related mathematical models. This paper presents a structured study of fixed point and common fixed point theorems in fuzzy metric spaces and intuitionistic fuzzy metric spaces. The work is based on the research direction described in the associated doctoral synopsis and is centered on the development and generalization of common fixed point results under new contractive conditions. The study reviews classical fixed point principles, the development of fuzzy metric spaces, intuitionistic fuzzy metric spaces, compatibility conditions, and recent generalized mappings. Particular attention is given to modified intuitionistic fuzzy metric structures and the possibility of obtaining common fixed point results under weaker or more flexible conditions. A theoretical research methodology is proposed consisting of systematic literature review, identification of research gaps, formulation of contractive conditions, rigorous mathematical proof, and application-oriented verification. The base research direction is motivated by recent work on common fixed point theorems in intuitionistic fuzzy metric spaces and their application to Volterra integral equations. The paper does not claim experimental results not contained in the supplied research synopsis; instead, it establishes a publication-oriented framework for the proposed mathematical investigation.

Keywords:- Fixed Point Theory, Fuzzy Metric Space, Intuitionistic Fuzzy Metric Space, Common Fixed Point, Contractive Mapping, Compatible Mapping, Coincidence Point, Nonlinear Analysis.

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I. INTRODUCTION

Fixed point theory is a fundamental branch of functional analysis concerned with points that remain invariant under a mapping. Its development is closely associated with successive approximation methods for establishing the existence of solutions of differential equations. Many nonlinear equations, optimization problems, variational problems, partial differential equations, optimal-control problems, and inverse problems can be formulated through fixed point equations.

Classical results such as Brouwer's Fixed Point Theorem and Banach's Contraction Principle provide important existence and uniqueness conditions. Banach's theorem states, in the classical setting, that a contraction mapping on a complete metric space has a unique fixed point. The

importance of this principle has led researchers to extend fixed point techniques to generalized metric structures.

Fuzzy set theory introduced a framework for representing vagueness by assigning membership grades in the interval $[0,1]$. Fuzzy metric spaces subsequently generalized classical distance concepts, allowing uncertainty to be incorporated into metric descriptions. Intuitionistic fuzzy structures further represent information through membership and non-membership degrees. These structures are useful when the available information is incomplete, uncertain, or not adequately represented by a single membership value.

The present research focuses on common fixed point theorems in fuzzy metric spaces and intuitionistic fuzzy metric spaces. The proposed direction includes the improvement and extension of existing results through new contractive conditions, the study of modified intuitionistic-fuzzy metric spaces, and investigation of applications of fixed point theorems. These objectives are directly aligned with the supplied doctoral synopsis.

II. LITERATURE REVIEW

The development of fixed point theory has progressed from classical metric spaces to increasingly generalized structures. Brouwer's theorem established a topological existence principle, while Banach's contraction principle provided a constructive framework based on contractions. Subsequent research introduced common fixed points, coincidence points, compatibility conditions, and generalized contraction requirements.

In fuzzy mathematics, Zadeh introduced fuzzy sets in 1965, providing the mathematical basis for representing degrees of membership. Kramosil and Michalek introduced the fuzzy metric space concept, and George and Veeramani later developed a modified formulation using continuous t -norms and established a useful Hausdorff topology. Grabiec extended classical fixed point principles to fuzzy metric spaces.

Intuitionistic fuzzy metric spaces extend fuzzy descriptions by considering both membership and non-membership. Atanassov introduced intuitionistic fuzzy sets, while Park developed intuitionistic fuzzy metric spaces using continuous t -norms and t -conorms. Subsequent researchers investigated compatible, sub-compatible, weakly compatible, faintly compatible, and related classes of mappings in generalized fuzzy settings.

Table 1: Selected literature relevant to fixed point theory, fuzzy metric spaces and intuitionistic fuzzy metric spaces.

Author / Year	Area / Method	Major Contribution
Banach (1932)	Contraction Principle	Existence and uniqueness of a fixed point in complete metric spaces.
Zadeh (1965)	Fuzzy Sets	Introduced membership grades for representing vagueness.
Nadler (1969)	Multivalued contractions	Extended contraction ideas to multivalued mappings.
Kramosil & Michalek (1975)	Fuzzy metric spaces	Introduced the fuzzy metric space concept.
Jungck (1976/1986)	Commuting/compatible maps	Developed common fixed point results for generalized mappings.
George & Veeramani (1994)	Fuzzy metric spaces	Modified fuzzy metric formulation using t-norms.
Atanassov (1986)	Intuitionistic fuzzy sets	Introduced intuitionistic membership and non-membership representation.
Park (2004)	Intuitionistic fuzzy metric spaces	Developed intuitionistic fuzzy metric spaces.
Ben et al. (2023)	Common fixed points	Studied common fixed point theorems with an application to Volterra integral equations.
Muraliraj et al. (2024)	Modified fuzzy metric spaces	Investigated fixed point theorems in modular revised fuzzy metric spaces.

The reviewed literature indicates that fixed point theory has continuously moved toward weaker assumptions, generalized contractive conditions, and broader classes of mappings. The supplied synopsis identifies a research opportunity in developing reliable and compatible fixed point theorems in fuzzy and intuitionistic fuzzy metric settings, particularly through new contractive conditions and modified structures.

III. MATHEMATICAL PRELIMINARIES

Let X be a non-empty set and $T: X \rightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of T if $T(x) = x$. If T and S are two mappings from X into itself, x is a coincidence point when $T(x) = S(x)$. If, in addition, $T(x) = S(x) = x$, then x is a common fixed point of T and S .

A fuzzy set assigns to each element a membership value in $[0,1]$. In fuzzy metric theory, the degree to which two points are considered close is represented through a fuzzy metric function rather than a single crisp distance. The construction allows uncertainty in the concept of distance to be incorporated into fixed point analysis.

An intuitionistic fuzzy framework supplements membership information with non-membership information. The associated metric structure can therefore describe both the degree of nearness and the degree of non-nearness. Continuous t -norms and t -conorms are important operations in the formulation of intuitionistic fuzzy metric spaces.

Compatibility conditions are frequently used in common fixed point theory. The literature reviewed in the synopsis includes compatible mappings, compatible mappings of type (P), semi-compatible mappings, sub-compatible mappings, occasionally weakly compatible mappings, faintly compatible mappings, and related generalized conditions.

IV. BASE RESEARCH DIRECTION

The foundational research direction selected in the synopsis is the paper entitled “Common fixed point theorems in intuitionistic fuzzy metric spaces with an application for Volterra integral equations” by A. Leila Ben et al. (2023), published in *Communications in Nonlinear Science and Numerical Simulation*, Volume 127. The study first establishes a common fixed point theorem in L -fuzzy metric spaces, introduces compatible mappings of type (P), examines their relationships with other compatibility notions, and establishes coincidence and fixed point results for multiple mappings.

This work provides a theoretical starting point for the proposed research. The intended extension is not to reproduce the base result, but to investigate new contractive assumptions and modified intuitionistic-fuzzy metric structures. The research will examine whether common fixed point conclusions can be obtained under conditions weaker, different, or more general than those already established.

V. PROPOSED METHODOLOGY

The proposed methodology follows a mathematical and analytical research design. The principal stages are literature review, research-gap identification, theorem formulation, proof development, verification through examples or counterexamples where appropriate, and application-oriented investigation.

- Conduct a comprehensive review of fixed point results in metric, fuzzy metric, and intuitionistic fuzzy metric spaces.
- Classify existing contractive conditions and compatibility assumptions used for common fixed point results.
- Identify assumptions that can potentially be weakened, replaced, or generalized.
- Formulate new contractive conditions suitable for intuitionistic fuzzy metric structures.
- Develop candidate common fixed point and coincidence point theorems for pairs or multiple mappings.
- Establish the proposed results through rigorous mathematical proof using direct arguments, contradiction, iterative sequences, and related techniques.
- Construct suitable examples to demonstrate the applicability of the proposed hypotheses and distinguish the results from existing theorems.
- Investigate applications such as integral equations or other nonlinear problems where the fixed point results can provide existence and uniqueness conditions.

A. Proposed Research Framework

The research framework can be represented as a sequence of theoretical stages: existing theory → research gap → modified contractive condition → theorem formulation → mathematical proof → examples/counterexamples → comparison with existing results → application.

Table 2: Proposed research framework.

Stage	Input	Expected Activity	Output
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1. Literature Review	Existing papers and theorems	Study assumptions, mappings and conclusions	Knowledge base
2. Gap Identification	Limitations of existing results	Find restrictive or unexplored conditions	Research problem
3. Condition Design	Fuzzy/intuitionistic structure	Formulate generalized contraction	Candidate condition
4. Theorem Development	Candidate condition	Construct fixed point statement	Proposed theorem
5. Proof	Theorem statement	Rigorous mathematical analysis	Validated theorem
6. Verification	Examples / special cases	Test assumptions and implications	Illustrative results
7. Application	Validated theorem	Apply to nonlinear/integral problem	Application result

B. Proposed General Contractive Perspective

A central objective is to investigate contractive conditions that generalize known fixed point principles in an intuitionistic fuzzy environment. Rather than assuming that a classical contraction transfers directly to the fuzzy setting, the proposed research will formulate conditions in terms of the relevant fuzzy membership, non-membership, t-norm, t-conorm, and metric parameters.

A generic research form may be expressed as a contractive relation involving mappings T and S and a control function ϕ , where the exact form of the inequality will be determined after the detailed literature-gap analysis. The research will then investigate sufficient conditions for the existence and uniqueness of a point x satisfying $T(x)=S(x)=x$.

The important point is that the present paper treats this expression as a research framework rather than as a proved new theorem. The final theorem and its hypotheses must be established during the doctoral research through complete mathematical proofs.

C. Algorithmic Research Procedure

Input: A complete intuitionistic fuzzy metric structure and one or more self-mappings.

Output: A rigorously established fixed point/common fixed point result, when the required hypotheses hold.

Step 1: Define the underlying set, intuitionistic fuzzy metric structure, t-norm and t-conorm.

Step 2: Specify the mappings and the required compatibility or related mapping conditions.

Step 3: State the candidate contractive condition.

Step 4: Construct an appropriate iterative sequence or sequence of coincidence points.

Step 5: Use completeness and the intuitionistic fuzzy metric properties to investigate convergence.

Step 6: Prove that the limiting point satisfies the fixed point or common fixed point relation.

Step 7: Establish uniqueness when supported by the contractive assumptions.

Step 8: Compare the obtained result with selected existing theorems to identify the precise generalization.

Step 9: Develop an example or application demonstrating the significance of the result.

VI. EXPECTED RESULTS AND DISCUSSION

The supplied synopsis does not report completed numerical experiments or final mathematical theorems. Therefore, the present paper does not fabricate theorem statements, numerical performance values, or experimental outcomes. Instead, the expected research outcomes are formulated from the stated doctoral objectives.

Table 3: Expected outcomes derived from the doctoral research objectives.

Research Objective	Expected Outcome	Evaluation Basis
Common fixed points in fuzzy metric spaces	Improved or extended common fixed point results	Comparison with established theorems
Common fixed points in	New existence/uniqueness conditions	Rigorous mathematical

intuitionistic fuzzy metric spaces		proof
New contractive conditions	Generalized fixed point framework	Reduction to or comparison with known conditions
Modified intuitionistic- fuzzy spaces	New theorems under modified structures	Consistency and examples
Applications	Use of fixed point results in suitable mathematical problems	Existence/uniqueness analysis

A. Research Gap

The literature review indicates a broad progression from classical contractions to generalized mappings and fuzzy structures. The principal gap identified for the proposed study is the continued need for fixed point and common fixed point results under new or relaxed contractive conditions in intuitionistic fuzzy metric spaces. The synopsis specifically targets modified intuitionistic-fuzzy metric spaces and new contractive conditions. The research therefore aims to contribute results that extend existing theorems while maintaining mathematically meaningful assumptions.

B. Significance of the Proposed Work

The significance of the proposed research is primarily theoretical. New fixed point results can enlarge the toolbox available for nonlinear analysis and provide sufficient conditions for existence and uniqueness of solutions to mathematical problems. The application-oriented component may connect the abstract results with integral equations and other nonlinear models. The base research direction also indicates the relevance of Volterra integral equations as a possible application domain.

VII. APPLICATION PERSPECTIVE

Fixed point methods are useful when a mathematical problem can be reformulated as $x=T(x)$. In an intuitionistic fuzzy environment, the generalized metric can accommodate uncertainty associated with the mathematical description. A future application of the proposed results may

proceed by converting a suitable nonlinear or integral equation into an operator equation and then verifying the hypotheses of an established common fixed point theorem.

For example, in a Volterra-type integral equation, the equation can be represented through an associated integral operator. If the operator satisfies an appropriate contraction or generalized contraction in the selected space, a fixed point theorem can be used to establish existence, and under stronger assumptions, uniqueness. The exact application will depend on the theorem ultimately proved during the research.

VIII. COMPARATIVE ANALYSIS OF THEORETICAL SETTINGS

Table 4: Comparison of theoretical settings relevant to the proposed study.

Setting	Main Feature	Fixed Point Focus	Research Relevance
Classical Metric Space	Crisp distance	Banach/Brouwer and generalizations	Foundation
Fuzzy Metric Space	Degree-based nearness	Fixed/common fixed points	Generalized distance
Intuitionistic Fuzzy Metric Space	Membership and non-membership information	Common fixed points under generalized conditions	Primary research setting
L-Fuzzy Metric Space	Lattice-valued/generalized structure	Generalized common fixed points	Base-paper direction
Modified IF Metric Structure	Extended or altered fuzzy conditions	New contractive results	Proposed research direction

IX. CONCLUSION

This paper presented a structured research framework for the study of fixed point and common fixed point theorems in fuzzy metric spaces and intuitionistic fuzzy metric spaces. The discussion reviewed the development from classical fixed point theory to fuzzy and intuitionistic fuzzy

structures, together with compatibility concepts and generalized mapping conditions. The selected base research direction on common fixed point theorems and Volterra integral equations provides a strong foundation for the proposed investigation.

The principal contribution planned in the doctoral work is the development of improved and generalized fixed point results using new contractive conditions and modified intuitionistic-fuzzy metric spaces. The proposed methodology emphasizes rigorous mathematical proofs, examples, comparison with existing results, and application-oriented analysis. Because the supplied synopsis does not contain completed theorem proofs or experimental results, those elements are intentionally presented as future research outcomes rather than reported findings.

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